XXVI.

SPHERICAL CONICS.

THE THESIS OF A CANDIDATE FOR MATHEMATICAL HONORS CONFERRED WITH THE DEGREE OF A.B., AT HARVARD COLLEGE, AT COMMENCEMENT, 1877.

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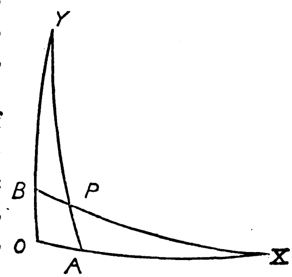
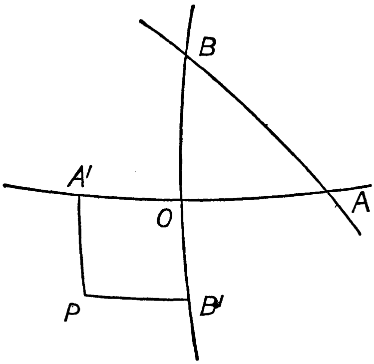
1. IT is convenient in dealing with spherical curves to have a system of spherical co-ordinates similar to plane co-ordinates. Such a system can be constructed as follows: Through the origin of plane co-ordinates draw a sphere tangent to the plane with a radius equal to unity, and project the plane axes upon the sphere by drawing lines from each point to the centre. The plane axes will thus be projected into semicircles having their extremities upon the circle of which the origin is the pole. (By circles and arcs I shall always mean great circles and their arcs, unless it is otherwise specified.) Every point on the plane will be represented by a point on the hemisphere, and this latter point can be referred to the projections of the plane axes as spherical axes. The plane co-ordinates of a point, measured on the axes, will be projected into arcs of the spherical axes, whose tangents are equal to the plane co-ordinates. The tangents of the arcs are therefore taken as spherical co-ordinates instead of the arcs themselves. Moreover, since all lines parallel to the plane axes meet them at infinity, such lines will be projected into arcs passing through the extremities of the spherical axes. Therefore, to find the spherical co-ordinates of a point on the sphere, draw arcs through the point and the extremities of the spherical axes, and take the tangents of the intercepts of these arcs as co-ordinates. Thus the co-ordinates of P are tan OA $= x$, tan OB $= y$. The spherical axes may be inclined at any angle, but I shall confine myself to rectangular axes.



2. It is evident that, in this system of co-ordinates, a straight line in the plane will be projected into an arc whose equation will be of the first degree in spherical co-ordinates, and that in general a locus of the nth degree in the plane will be represented by one of the nth degree in spherical co-ordinates. The equation of a great circle may then be written



$$Ax + By = 1. \qquad (a)$$

Let P be the pole of this circle; and let the co-ordinates of P be tan OA′ and tan OB′. But OA′ = OA — 90°, OB′ = OB — 90°. Hence, the co-ordinates of the pole are $-A$ and $-B$; that is, when the equation of a circle is written in the form (a), the co-ordinates of its pole are the negatives of the coefficients of x and y. This principle is of great utility in finding distances and equations on the sphere. The equation of a circle can also be written $y = mx + n$; and from this it can be deduced, as in plane co-ordinates, that the equation of an arc through a point $x'y'$ is

$$y - y' = m(x - x'),$$

and of that through two points is

$$\frac{y - y'}{x - x'} = \frac{y'' - y'}{x'' - x'}.$$

3. To find the distance between two points $x'y'$ and $x''y''$ on the sphere. The formulas for this, being well known, may be simply written out for reference. If ϱ denotes the distance, they are

$$\cos \varrho = \frac{1 + x'x'' + y'y''}{\pm [(1 + x'^2 + y'^2)(1 + x''^2 + y''^2)]^{\frac{1}{2}}},$$

$$\sin \varrho = \pm \left[\frac{(x'' - x')^2 + (y'' - y')^2 + (x'y'' - x''y')^2}{(1 + x'^2 + y'^2)(1 + x''^2 + y''^2)}\right]^{\frac{1}{2}},$$

$$\tan \varrho = \frac{\pm [(x'' - x')^2 + (y'' - y')^2 + (x'y'' - x''y')^2]^{\frac{1}{2}}}{1 + x'x'' + y'y''}.$$

4. A spherical conic is the intersection of a unit-sphere with a cone of the second degree, whose vertex is at the centre of the sphere. The arcs in which the cyclic planes of the cone cut the sphere are called

the cyclic arcs of the conic. Since the cone is double, it will cut the sphere in two closed curves; and we therefore name the conic differently according to the hemisphere considered. If the sphere be divided by the principal plane of the cone, it gives a closed curve whose centre will be the pole of the dividing circle, and whose principal diameters will be the arcs of the greatest and least sections of the cone. The cyclic arcs will intersect at the points where the arc of greatest section meets the dividing circle, and will be symmetrical with reference to the curve. This form of conic is a *Spherical Ellipse.*

If the sphere be divided by the plane of least section of the cone, the conic will consist of two branches. Its centre will be the pole of the dividing circle, and its principal diameters will be the arcs made by the plane of greatest section of the cone and the principal plane. The cyclic arcs meet only once, and that at the centre. This curve is the *Spherical Hyperbola,* and it will be found that its cyclic arcs have properties analogous to those of the asymptotes to the plane hyperbola.

If, again, the sphere be bisected by a plane perpendicular to the two already mentioned, there is still a third form of spherical conic, having its centre at the pole of the bisecting circle. There is, properly speaking, as might be expected from the method of projection used, no spherical parabola. If a plane parabola be projected upon a sphere, points at infinity are projected, and the spherical parabola is merely an ellipse or an hyperbola. The conic of which the major axis is a quadrant has, however, the closest analogy to the Parabola.

5. A spherical conic may also be defined as the locus of an equation of the second degree in spherical co-ordinates. The general equation is

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.$$

This can be transformed to the centre as origin; and, if we choose the principal diameters as axes, it can be reduced to the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

The equation for determining the centre is a cubic, and this shows that a spherical conic has three centres. We are thus led in another way to the results arrived at in Art. 4.

This method of reducing the general equation is, however, on account of the complex formulas for transformation of spherical co-ordinates, long and tedious. It is better, therefore, to derive the equation referred to the centre from the central equation of plane conics.

By the principles explained in Art. 1, the central equation may be written $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where a and b are the tangents of the principal semi-diameters.

6. Certain properties of the spherical conics follow immediately from the quaternion equation of the cone, and it may be well to introduce the equation here. The general form of the equation, as given by Tait, is

$$S\rho\phi\rho = 0.$$

A particular form of this is

$$\rho^2 - S\alpha\rho S\beta\rho = 0,$$

where α and β are perpendicular to the cyclic planes.

7. To find the equations of tangent and polar arcs.

The equation of a tangent to a spherical conic is found, as for the plane curve, by determining the value of $\frac{y'' - y'}{x'' - x'}$, when $x'' = x'$ and $y'' = y'$, and substituting and reducing. The equation is $\frac{xx'}{a^2} + \frac{yy'}{b^2} = 1$. Since this represents a tangent when $x'y'$ is on the curve, it must, from the symmetry of the equation, represent the arc on which lie the points of contact of tangents from $x'y'$, if $x'y'$ is not on the curve; that is, it is the polar of $x'y'$. (When $x'y'$ is a pole with respect to the conic, I shall call it a *conic* pole, to distinguish it from the ordinary pole of circles.)

The symmetry of the equation shows that if $x''y''$ lies on the polar of $x'y'$, then $x'y'$ lies on the polar of $x''y''$. There are many properties of polars to spherical conics similar to those of plane geometry.

8. We can now find the equation of the locus of the extremity of a quadrant moving at right angles to the given conic; that is, the locus of the pole of the tangent. Thus

$$x = -\frac{x'}{a^2}, y = -\frac{y'}{b^2}, \text{ or } x' = -a^2x, y' = -b^2y,$$

$$\text{but } \frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 1\,;$$

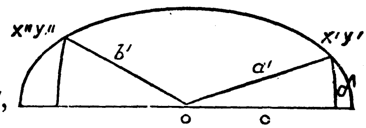
$$\therefore \quad a^2x^2 + b^2y^2 = 1$$

is the required equation. This is a conic the tangents of whose semi-axes are $\frac{1}{a}$ and $\frac{1}{b}$: its semi-axes are therefore the complements of those

of the given conic. This conic is called the *supplementary conic* of the given one. It can be combined with the given conic so as to simplify the solution of many questions in spherical conics.

9. *Conjugate Diameters.* These are related to each other as in plane conics; that is, the diameter conjugate to the one through $x'y'$ contains the conic pole of the one through $x'y'$, and *vice versa.* Its equation is therefore $\frac{xx'}{a^2} + \frac{yy'}{b^2} = 0$. Its extremity $x''y''$ is found, as in plane conics, to be such that

$$\frac{x''}{a} = \pm \frac{y'}{b}, \frac{y''}{b} = \mp \frac{x'}{a}.$$



To find the lengths of a' and b', any two conjugate semi-diameters.

$$\cos a' = \cos c \cos \delta = \frac{1}{(1 + x'^2 + y'^2)^{\frac{1}{2}}};$$

$$\tan^2 a' = x'^2 + y'^2 = b^2 + \frac{a^2 - b^2}{a^2} x'^2;$$

and $$\tan^2 b' = x''^2 + y''^2 = a^2 - \frac{a^2 - b^2}{a^2} x'^2;$$

$$\therefore \quad \tan^2 a' + \tan^2 b' = a^2 + b^2 = \text{constant}.$$

This might also be inferred from the corresponding properties of plane conics, by the principles laid down in Art. 1.

10. To find the perpendicular distance from the centre on a tangent.

The trigonometric tangent of the perpendicular from the centre is the cotangent of the distance of the pole from the centre. Calling the perpendicular from the centre p, we have

$$\cot p = \sqrt{\frac{x'^2}{a^4} + \frac{y'^2}{b^4}} = \frac{\sqrt{\left(\frac{b^2x'^2}{a^2} + \frac{a^2y'^2}{b^2}\right)}}{ab} = \frac{\tan b'}{ab},$$

$$\therefore \quad \tan p = \frac{ab}{\tan b'}.$$

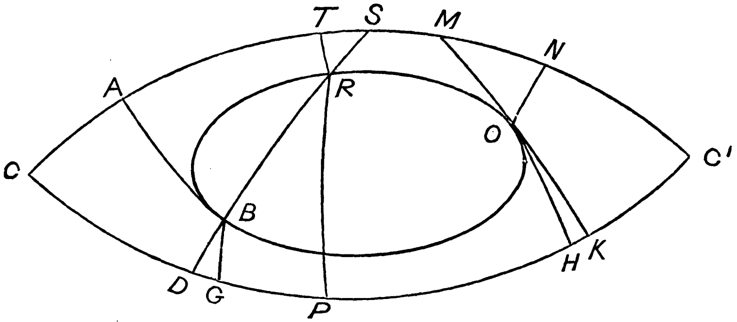
11. There are some curious properties of conics with reference to the cyclic arcs. (α) We have from the quaternion equation of the cone

$$\cos \theta \cos \theta' = k,$$

or, since α and β are perpendicular to the cyclic planes,

$$\sin \varrho \sin \varrho' = k,$$

where ϱ and ϱ' denote arcs from any point of the conic perpendicular to the cyclic arcs.



(β) If a great circle cut a spherical conic, the parts of it between the points of intersection and the respective cyclic arcs are equal. For

$$\sin \text{D} = \frac{\sin \text{BG}}{\sin \text{BD}} = \frac{\sin \text{RP}}{\sin \text{RD}}, \quad \sin \text{S} = \frac{\sin \text{RT}}{\sin \text{RS}} = \frac{\sin \text{AB}}{\sin \text{BS}};$$

and therefore, by (α),

$$\sin \text{BS} \sin \text{DB} = \sin \text{RS} \sin \text{DR},$$

or $$\sin (\text{BR} + \text{RS}) \sin \text{DB} = \sin \text{RS} \sin (\text{DB} + \text{BR});$$

$$\therefore \quad \sin \text{DB} \cos \text{RS} = \sin \text{RS} \cos \text{DB},$$

$$\therefore \quad \text{DB} = \text{RS}.$$

I shall also insert a quaternion proof, as given by Tait. If a conic be cut by a plane whose equation is $S\gamma\varrho = 0$, the intersections of this with the cyclic planes are $V\alpha\gamma$ and $V\beta\gamma$. Then, since a point of the curve can be reached by moving in the directions of these intersections, we may write

$$\varrho = xUV\alpha\gamma + yUV\beta\gamma,$$

$$S\alpha\varrho = yS\alpha UV\beta\gamma,$$

$$S\beta\varrho = xS\beta UV\alpha\gamma.$$

$\therefore\ \varrho^2 - S\alpha\varrho S\beta\varrho = 0$ may be written

$$x^2 + y^2 + Axy = 0,$$

where A is a scalar function of α, β, and γ only. The form of the equation shows that any two values of x and y can be interchanged. This, then, establishes the theorem.

If the cutting arc becomes a tangent, the parts intercepted between the point of contact and the cyclic arcs are equal.

12. From the two properties announced, we can deduce another. The area of the spherical triangle formed by a tangent and the cyclic arcs is constant.

$$\sin \text{ON} = \sin \text{OM} \sin \text{M},$$

$$\sin \text{OH} = \sin \text{OK} (\sin \text{OKH} = \sin \text{OK}C').$$

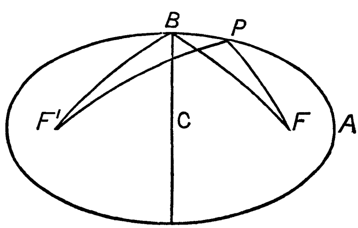
$$\therefore \quad \sin^2 \text{OK} \sin \text{M} \sin \text{K} = \text{constant}.$$

But $\quad \cos C' = -\cos(\text{M} + \text{K}) - 2 \sin^2 \text{OK} \sin \text{M} \sin \text{K}.$

Hence, since C' is constant for a given conic, $\text{M} + \text{K}$ is constant, and therefore the area of the triangle is constant, for it equals $C' + \text{M} + \text{K} - 180°$.

We may then define a spherical conic as the envelope of the base of a spherical triangle, of which the vertex, the vertical angle, and the area are given. The arcs forming the vertical angle are the cyclic arcs of the conic.

13. This property is also true of the supplementary conic; and therefore, remembering the relation existing between arcs and their poles, we may define a spherical conic as the locus of the vertex of a spherical triangle, of which the base is given in length and position, and of which the sum of the sides is also given. The extremities of the base are the *foci* of the conic; and we now wish to determine their position. It is evident that they are the poles of the supplementary cyclic arcs. Since, by Art. 11 (β), a conic is symmetrical with respect to its cyclic arcs, their poles must lie on an axis at equal distances from the centre; and, since the axes of conics and those of supplementary conics are parts of the same circles, the foci of a conic must lie on an axis of that conic, at equal distances from the centre. It can be shown that this axis is the major axis. Let F and F′ be the foci; then, as the sum of the sides of a spherical triangle is greater than the base, the foci fall inside the conic. When P is at A,



$$\text{F}'\text{A} + \text{FA} = \text{constant} = 2\,\text{CA};$$

and, when at B,

$$\text{F}'\text{B} + \text{FB} = 2\,\text{FB} = 2\text{CA};$$

$$\therefore \quad \text{CA} = \text{FB}:$$

$$\cos \text{FB} = \cos \text{BC} \cos \text{CF} = \cos \text{CA},$$

$$\therefore \quad \cos \text{CA} < \cos \text{BC}, \qquad \therefore \quad \text{CA} > \text{BC}.$$

14. The equation of the conic may be determined by finding the locus of the vertex of a triangle, when the base and the sum of the sides are given.

Suppose $A + B = 2\alpha$; then

$$\cos A \cos B - \sin A \sin B = \cos 2\alpha,$$

$$(\cos A \cos B - \cos 2\alpha)^2 = (1 - \cos^2 A)(1 - \cos^2 B),$$

$$\cos^2 A + \cos^2 B - 2 \cos 2\alpha \cos A \cos B = \sin^2 2\alpha;$$

but $\cos 2\alpha = 1 - 2 \sin^2 \alpha$, $\sin 2\alpha = 2 \sin \alpha \cos \alpha$;

$$\therefore \quad (\cos A - \cos B)^2 + 4 \sin^2 \alpha \cos A \cos B = 4 \sin^2 \alpha \cos^2 \alpha.$$

Let $\text{FC} = c$, and use the formulas of Art. 3; then

$$x^2 \tan^2 c + (1 - x^2 \tan^2 c) \sin^2 \alpha = (1 + x^2 + y^2) \sin^2 \alpha \cos^2 \alpha \sec^2 c.$$

But $\cos c = \dfrac{\cos a}{\cos \beta}$, $\sec^2 c = \dfrac{\cos^2 \beta}{\cos^2 a}$, $\tan^2 c = \dfrac{\cos^2 \beta - \cos^2 a}{\cos^2 a}$;

and hence, by easy reductions,

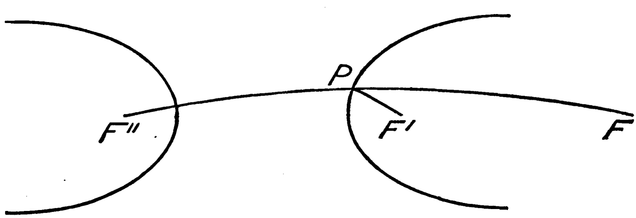
$$x^2 \cos^2 \alpha \sin^2 \beta + y^2 \sin^2 \alpha \cos^2 \beta = \sin^2 \alpha \sin^2 \beta,$$

or

$$x^2 \cot^2 \alpha + y^2 \cot^2 \beta = 1,$$

which is the same form of equation as that given in Art. 5.

15. If the sphere be so divided as to make a spherical hyperbola, then the locus becomes the vertex of a triangle whose base and also the difference of its sides are given.



$$\text{F}''\text{P} + \text{PF} = 180^\circ, \quad \text{F}'\text{P} + \text{PF} = \text{constant};$$

$$\therefore \quad \text{F}''\text{P} - \text{F}'\text{P} = \text{constant}.$$

As the spherical hyperbola is simply made up of the halves of two equal ellipses, it is not necessary to deal with it separately; for whatever is true of the ellipse is true of the hyperbola.

16. There are certain relations between α, β, and c, which enable us to reduce some forms of equations.

$$\cos c = \frac{\cos \alpha}{\cos \beta}, \quad \tan \alpha = a, \quad \tan \beta = b;$$

$$\therefore \quad \frac{\sin^2 c}{\sin^2 \alpha} = \frac{\frac{1}{b^2+1} - \frac{1}{a^2+1}}{\frac{a^2}{(b^2+1)(a^2+1)}} = \frac{a^2 - b^2}{a^2} = \varepsilon_1^2,$$

$$\frac{\tan^2 c}{\tan^2 \alpha} = \frac{\frac{1}{b^2+1} - \frac{1}{a^2+1}}{\frac{a^2}{a^2+1}} = \frac{a^2 - b^2}{a^2(1+b^2)} = \varepsilon^2,$$

$$\frac{\sin^2 2c}{\sin^2 2\alpha} = \frac{(a^2+1)(b^2+1) - (b^2+1)^2}{a^2} = \frac{(a^2-b^2)(1+b^2)}{a^2} = \varepsilon'^2.$$

17. By Art. 11, $\sin \rho \sin \rho' = k$; and from this can be proved a property of the foci similar to one in plane conics. Since the foci of a conic are the poles of the cyclic arcs of the supplementary conic, the distances of the foci from any tangent are equal to the distances of the corresponding point of the supplementary conic from its cyclic arcs. Hence, if δ and δ_1 denote the distances of the foci from a tangent, $\sin \delta \sin \delta_1 =$ constant.

This constant can be determined as follows: — Using the pole of the tangent as in Art. 10, we have

$$\sin \delta = \frac{ab^2(a - \varepsilon x')}{[(a^4b^4 + b^4x'^2 + a^4y'^2)(1 + a^2\varepsilon^2)]^{\frac{1}{2}}};$$

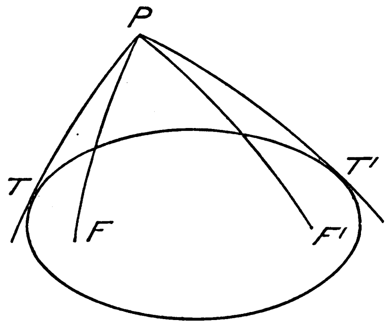
$$a^4b^4 + b^4x'^2 + a^4y'^2 = a^2b^2(a^2b^2 - \frac{a^2-b^2}{a^2}x'^2 + a^2),$$

$$\frac{a^2-b^2}{a^2} = \frac{\varepsilon^2(a^2+1)}{a\varepsilon^2+1}, \quad a^2(b^2+1) = \frac{a^2(a^2+1)}{a^2\varepsilon^2+1};$$

$$\therefore \quad \sin \delta = \frac{b(a - \varepsilon x')}{[(a^2 - \varepsilon^2x'^2)(a^2+1)]^{\frac{1}{2}}}, \quad \sin \delta_1 = \frac{b(a + \varepsilon x')}{[(a^2 - \varepsilon^2x'^2)(a^2+1)]^{\frac{1}{2}}},$$

$$\sin \delta \sin \delta_1 = \frac{b^2}{a^2+1}.$$

18. The two tangents drawn from any point to a spherical conic make equal angles with the arcs joining that point to the foci. The poles of PT and PT′ lie on the supplementary conic; the poles of PF and PF′ lie on the supplementary cyclic arcs. Moreover, all these poles lie on the circle of which P is the pole. Hence, by Art. 11 (β), the poles of PT and PF are the same distance apart as those of PT′ and PF′, and therefore the angles TPF and T′PF′ are equal.



When the point P is on the conic, this theorem becomes the following, which was one of the first discovered properties of spherical conics: — The two arcs from the foci to any point of the conic make equal angles with the tangent at that point. It follows immediately also that, if from any point of one of two confocal conics tangent arcs be drawn to the other, these tangents make equal angles with the tangent to the first conic at the given point.

19. The theorem of Art. 18 also proves that, if two confocal conics intersect, they intersect at right angles.

Since two confocal conics imply two supplementary concyclic conics, we see that, if a common tangent arc be drawn to two concyclic conics, the part intercepted between the two points of contact will be a quadrant. This is evident from the first part of this article, for this arc measures the right angle made by the two confocal conics.

20. We now come to an important theorem: — The projection of a spherical conic on a tangent plane to the sphere at one of the foci is a plane conic having the point of contact for a focus. Let $\varrho + \varrho' = 2\alpha$; then if $\theta =$ the angle which ϱ makes with the axis,

$$\cos \varrho' = \cos 2\alpha \cos \varrho + \sin 2\alpha \sin \varrho ;$$

but we also have by spherical trigonometry

$$\cos \varrho' = \cos \varrho \cos 2c + \sin \varrho \sin 2c \cos \theta,$$

$$\therefore \quad \tan \varrho = \frac{\cos 2c - \cos 2a}{\sin 2a - \sin 2c \cos \theta},$$

which is the polar equation of a spherical conic referred to a focus as pole. Since $\tan \varrho$ is the projection of ϱ on the plane, and θ remains the same in the plane, — being the angle of the tangents to the sphere

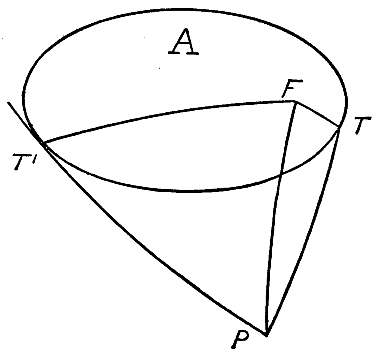
at the focus, — the equation shows that the projection on the plane is a conic with the same focus and an eccentricity equal to $\frac{\sin 2c}{\sin 2a}$.

The importance of this principle is due to the fact that it enables us to establish many properties of spherical conics with reference to a single focus from known properties of plane conics.

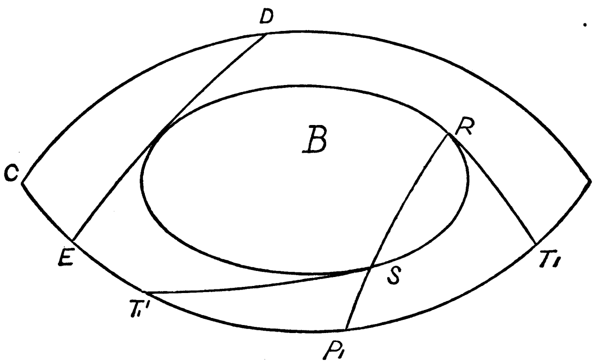
21. The angle subtended at the focus by any chord is bisected by the arc joining the focus to its conic pole. The spherical conic can be projected into a plane conic having the same focus, of which this property is true; but the angles of the plane conic at the focus are the measures of the angles of the spherical conic.

From this can be established a reciprocal property by the aid of the supplementary conic.

Two tangent arcs to a conic and the arc joining their points of contact cut the cyclic arc in three points, one of which bisects the distances between the other two.



Suppose A represents a conic and B its supplementary conic, and $\angle$T'FP $= \angle$TFP. The poles of FT', FP, and FT all lie on a cyclic arc of the supplementary conic. Let these poles be T'_1, P_1, and T_1 respectively; then $\overset{\frown}{T'_1P_1} = \overset{\frown}{P_1T_1}$. The poles of PT and PT' lie on the supplementary conic; denote them by R and S. Then R, S, and P_1 lie on the circle of which P is the pole. Moreover, T_1' and S lie on the circle of which



T' is the pole, and T_1 and R on that of which T is the pole; but the circles of which T' and T are the poles are tangents to the supplementary conic. Hence the theorem is established.

22. The directrix, or director arc, is the conic polar of the focus. We have, as a particular case of Art. 21: The arc joining the focus to the conic pole of any arc passing through the focus is perpendicular to the latter arc.

Also every tangent to a conic and the arc joining its point of contact to the conic pole of a cyclic arc meet that cyclic arc in two points a quadrant apart.

By the theory of projections, explained in Art. 21, we have: The arcs drawn from the focus of a conic to the point of intersection of two tangents, and to the point where the arc through the points of contact meets the director arc, are at right angles to each other; or, in other words, if any chord PP' cut the directrix in D, then FD is the external bisector of $\angle PFP'$.

Then, by the same reasoning as in Art. 21, the converse of this can be proved. The arc passing through the conic pole of a cyclic arc and through the point of intersection of two tangent arcs meets the cyclic arc at a point a quadrant distant from that at which the cyclic arc is met by the arc joining the points of contact of these tangents.

23. The angle subtended at the focus by the part cut off on a variable tangent by two fixed tangents is constant. This is a right angle, if the two fixed tangents intersect on the directrix. If through two fixed points on a conic two arcs be drawn intersecting in a third point of the conic, they intercept a constant segment on the cyclic arc. This is a quadrant, if the arc joining the two fixed points passes through the conic pole of the cyclic arc.

Also, from the corresponding property of plane conics, the sum of the cotangents of the segments of a focal chord is constant. Reciprocally: — If, from a point upon the cyclic arc, tangents be drawn to a conic, the sum of the trigonometric cotangents of the angles which they make with the cyclic arc is constant; for these angles are measured by the segments of the focal chord of the supplementary conic.

The rectangle under the tangents of the segments of a focal chord is proportional to the sum of the tangents of the segments.

If, from a point upon the cyclic arc, tangents be drawn to a conic, the product of the trigonometric tangents of the angles which they make with the cyclic arc is proportional to the sum of the tangents of the angles.

Many other properties and their reciprocals can in the same way be deduced from known properties of plane conics.

24. The directrix being the conic polar of the focus, its equation is

$$\frac{x \tan c}{a^2} = 1, \quad \text{or } \frac{\varepsilon}{a}\, x = 1.$$

From this it can be proved that the sines of the distances of any point of the conic from a focus and the corresponding directrix are in a constant ratio. By the formulas of Art. 3, if ϱ denotes the distance of any point $x'y'$ of the conic from the directrix, then

$$\sin \varrho = \frac{a \mp \varepsilon x'}{[(a^2 + \varepsilon^2)(1 + x'^2 + y'^2)]^{\frac{1}{2}}}.$$

If ϱ' denotes the focal distance of $x'y'$, then

$$\sin \varrho' = \frac{[(a\varepsilon \mp x')^2 + (1 + a^2\varepsilon^2)y'^2]^{\frac{1}{2}}}{[(1 + a^2\varepsilon^2)(1 + x'^2 + y'^2)]^{\frac{1}{2}}};$$

but

$$y'^2 = \frac{b^2}{a^2}\,(a^2 - x'^2),$$

$$= \frac{1 - \varepsilon^2}{1 + a^2\varepsilon^2}\,(a^2 - x'^2),$$

$$\therefore \quad \sin \varrho' = \frac{a \mp \varepsilon x'}{[(1 + a^2\varepsilon^2)(1 + x'^2 + y'^2)]^{\frac{1}{2}}},$$

$$\therefore \quad \frac{\sin \rho'}{\sin \rho} = \sqrt{\frac{a^2 + \varepsilon^2}{1 + a^2\varepsilon^2}}.$$

Then, by the method already used several times, we have the reciprocal: — In a conic, the sine of the angle which a tangent to the curve makes with the cyclic arc is in a constant ratio to the sine of the distance of this tangent arc from the conic pole of the cyclic arc.

25. The equation of an arc can be written in the form

$$x \cos \alpha + y \sin \alpha = \tan p\,;$$

and from this it can, as in plane conics, be shown that the perpendicular from the centre on a tangent satisfies the equation

$$\tan^2 p = a^2 \cos^2 \alpha + b^2 \sin^2 \alpha.$$

This property cannot, as in plane conics, be used to find the locus of the intersection of tangents at right angles to each other. The latter problem can best be solved by finding the cone which is the locus of the intersection of tangent planes to a given cone at right angles to each other.

If $S\rho\varphi\rho = 0$ be the equation of a cone, $S\varpi\varphi\rho = 0$ is the equation of a tangent plane. Suppose ϖ is the line of intersection, and α, β, and γ are three rectangular unit vectors; then for different sides of the cone $\rho = \varpi + x\alpha$, $\rho_1 = \varpi + y\beta$, $\rho_2 = \varpi + z\gamma$, and x, y, and z are eliminated by the fact that, when $\varpi + x\alpha$, &c., are substituted in the equation of the cone, the roots of the quadratics in x, y, and z are equal, because α, β, and γ are in the tangent planes. Substituting $\varpi + x\alpha$ for ρ in $S\rho\varphi\rho = 0$, we have

$$S(\varpi + x\alpha)\varphi(\varpi + x\alpha) = S\varpi\varphi\varpi + 2xS\alpha\varphi\varpi + x^2S\alpha\varphi\alpha = 0\,;$$

and, since the roots are equal,

$$S^2\alpha\varphi\varpi = S\varpi\varphi\varpi . S\alpha\varphi\alpha.$$

By introducing the other values of ρ, we get also

$$S^2\beta\varphi\varpi = S\varpi\varphi\varpi . S\beta\varphi\beta,$$

$$S^2\gamma\varphi\varpi = S\varpi\varphi\varpi . S\gamma\varphi\gamma.$$

$$\therefore\ S^2\alpha\varphi\varpi + S^2\beta\varphi\varpi + S^2\gamma\varphi\varpi = (S\alpha\varphi\alpha + S\beta\varphi\beta + S\gamma\varphi\gamma)S\varpi\varphi\varpi.$$

But it is known that

$$\varphi\varpi = \frac{iSi\varpi}{a^2} + \frac{jSj\varpi}{b^2} + \frac{kSk\varpi}{c^2}.$$

Hence we have

$$(\varphi\varpi)^2 = -\frac{S^2i\varpi}{a^4} - \frac{S^2j\varpi}{b^4} - \frac{S^2k\varpi}{c^4} = S\varpi\varphi^2\varpi,$$

$$S^2\alpha\varphi\varpi = \frac{S^2\alpha iS^2i\varpi}{a^4} + \frac{S^2\alpha jS^2j\varpi}{b^4} + \frac{S^2\alpha kS^2k\varpi}{c^4}$$

$$+ 2\left(\frac{S\alpha iS\alpha jSi\varpi Sj\varpi}{a^2b^2} + \frac{S\alpha jS\alpha kSj\varpi Sk\varpi}{b^2c^2} + \frac{S\alpha kS\alpha iSk\varpi Si\varpi}{c^2a^2}\right),$$

with similar expressions for $S^2\beta\varphi\varpi$ and $S^2\gamma\varphi\varpi$. Then remembering that

$$S^2\alpha i + S^2\beta i + S^2\gamma i = -i^2 = 1,$$

$$S\alpha iS\alpha j + S\alpha jS\alpha k + S\alpha kS\alpha i = 0,$$

we have

$$S^2\alpha\varphi\varpi + S^2\beta\varphi\varpi + S^2\gamma\varphi\varpi = -(\varphi\varpi)^2 = -S\varpi\varphi^2\varpi.$$

$$\text{Also } S\alpha\varphi\alpha = \frac{S^2\alpha i}{a^2} + \frac{S^2\alpha j}{b^2} + \frac{S^2\alpha k}{c^2}\,;$$

$$\therefore\ S\alpha\varphi\alpha + S\beta\varphi\beta + S\gamma\varphi\gamma = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}.$$

The equation of the cone sought becomes then

$$-S\varpi\phi^2\varpi = \left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right)S\varpi\phi\varpi;$$

but
$$S\varpi\phi\varpi = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2},$$

and
$$-S\varpi\phi^2\varpi = \frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4};$$

$$\therefore \quad \left(\frac{1}{b^2} + \frac{1}{c^2}\right)\frac{x^2}{a^2} + \left(\frac{1}{a^2} + \frac{1}{c^2}\right)\frac{y^2}{b^2} + \left(\frac{1}{a^2} + \frac{1}{b^2}\right)\frac{z^2}{c^2} = 0, \text{ or}$$

$$(b^2 + c^2)x^2 + (a^2 + c^2)y^2 + (a^2 + b^2)z^2 = 0, \qquad (1)$$

is the equation of the required cone.

Now the equation of the supplementary cone being $S\tau\phi^{-1}\tau = 0$, where $\tau = \phi\rho$, its equation can be written

$$a^2x^2 + b^2y^2 + c^2z^2 = 0. \qquad (2)$$

In any cone, as $Mx^2 + Ny^2 + Pz^2 = 0$, the equation of the cyclic planes containing the axis of x, for example, is

$$y^2(N - M) + z^2(P - M) = 0,$$

so that (1) and (2) have the same cyclic planes. From this it follows that the locus of the intersection of two tangents to a spherical conic which meet at right angles is a conic concyclic with the conic supplementary to the given conic. Reciprocally: — Since the poles of these tangents are on the supplementary conic a quadrant apart, it results that the locus of the pole of a chord of 90° is a conic concyclic with the given conic, and the chord envelopes a conic confocal with the conic supplementary to the given one.

26. The locus of the intersection of tangents at the extremities of conjugate diameters can also be found. The equations of the tangents are

$$\frac{xx'}{a^2} + \frac{yy'}{b^2} = 1,$$

$$\frac{xx''}{a^2} + \frac{yy''}{b^2} = 1, \text{ or, by Art. 9, } \frac{xy'}{ab} - \frac{x'y}{ab} = 1;$$

then, squaring and adding,

$$\frac{x^2}{a^2}\left(\frac{x'^2}{a^2} + \frac{y'^2}{b^2}\right) + \frac{y^2}{b^2}\left(\frac{x'^2}{a^2} + \frac{y'^2}{b^2}\right) = 2,$$

$$\text{or } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 2,$$

which is a spherical conic.

27. Several additional properties of cyclic arcs can also be stated. First, to find their equation. Their poles are at the same distance from the centre on the axis of y as the foci of the supplementary conic are from its centre. Suppose α, β, c, and α', β', c', are the semi-diameters and focal distances of the two conics, then

$$\alpha' + \beta = \frac{\pi}{2}, \quad \alpha + \beta' = \frac{\pi}{2},$$

$$\cos c' = \frac{\cos a'}{\cos \beta'} = \frac{\sin \beta}{\sin a},$$

$$\tan^2 c' = \frac{a^2 - b^2}{b^2(a^2 + 1)}.$$

Hence, the co-ordinates of the poles of the cyclic arcs are

$$0 \text{ and } \pm \sqrt{\frac{a^2 - b^2}{b^2(a^2 + 1)}};$$

and the equation of the arcs can (Art. 2) be written

$$y = \mp b \sqrt{\frac{a^2 + 1}{a^2 - b^2}}.$$

Let $2\theta =$ the angle between the cyclic arcs; then $2\theta + 2c' = \pi$. Hence,

$$\sin \theta = \cos c' = \frac{\sin \beta}{\sin a},$$

so that the angle of the cyclic arcs is equal to the angle subtended by the minor axis at either focus.

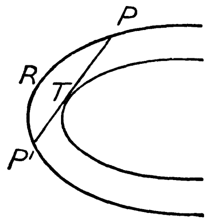
28. Every tangent to a spherical conic cuts the cyclic arcs in two points, such that the product of the trigonometric tangents of the halves of the arcs lying between these points and the point of intersection of the cyclic arcs is constant. It is known by spherical trigonometry that (see fig. B, Art. 21)

$$\tan(\text{area CED}) = \frac{\tan \frac{1}{2}\text{CD} \tan \frac{1}{2}\text{CE} \sin \text{C}}{1 + \tan \frac{1}{2}\text{CD} \tan \frac{1}{2}\text{CE} \cos \text{C}}.$$

$$\therefore \text{(by Art. 12)} \tan \tfrac{1}{2}\text{CD} \tan \tfrac{1}{2}\text{CE} = \text{constant}.$$

If arcs be drawn from any point of a spherical conic to the foci, the product of the tangents of the halves of the angles made by these arcs with the major axis is constant. For the intersection of the supplementary cyclic arcs is the pole of the major axis, and the poles of the arcs drawn from the foci lie on these cyclic arcs, at the extremities of a tangent to the supplementary conic. If the angles be measured in opposite directions, the ratio of the tangents of the semi-angles is constant; and a similar modification may be made in the reciprocal theorem.

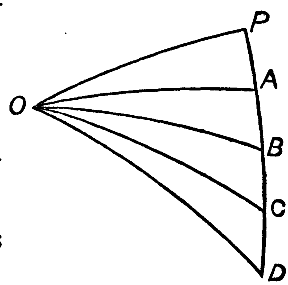
29. If a tangent be drawn to the inner of two concyclic conics, the parts included between the point of contact and the outer curve are equal. This is an immediate consequence of Arts. 11 and 12. Then, by the method of infinitesimals used in plane conics (Salmon's Conic Sections, § 396), the area included between the tangent and the outer curve is constant, as the point of contact moves along the inner curve.



The conics supplementary to concyclic conics are confocal. The pole T′ of PP′ lies upon the outer conic, and, if from T′ tangents be drawn to the inner conic, these tangents measure the angles which the tangents at P and P′ make with PP′. Moreover, the curve between the points of contact of tangents from successive positions of T′ measures the infinitely small angle made by consecutive tangents along the curve PRP′. But the sum of these angles with those at P and P′ mentioned above is constant. Hence this theorem follows: — If, from a point on the outer of two confocal conics, tangents be drawn to the inner one, the sum of these tangents and of the concave part of the curve included between them is constant.

30. I shall now give some principles of conic poles and polars with reference to spherical conics. The spherical anharmonic ratio is

$$\frac{\sin AD \sin BC}{\sin AB \sin CD}$$



To prove this ratio constant for any given pencil,

$$\sin AD = \frac{\sin AOD \sin OA}{\sin ADO} = \frac{\sin AOD \sin OA \sin OD}{\sin OP};$$

and, obtaining corresponding values for sin BC, &c., we have

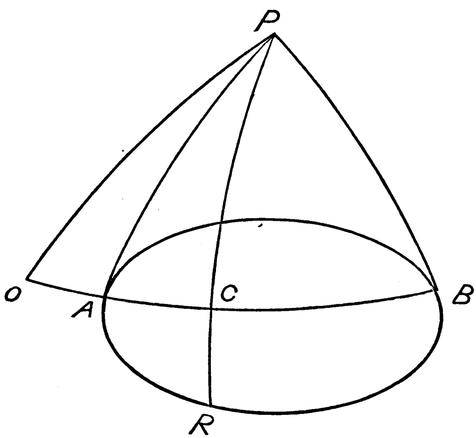
$$\frac{\sin \text{AD} \sin \text{BC}}{\sin \text{AB} \sin \text{CD}} = \frac{\sin \text{AOD} \sin \text{BOC}}{\sin \text{AOB} \sin \text{COD}};$$

so that the ratio depends only on the angles at O, and is constant for any given pencil.

31. To prove that an arc drawn from a conic pole O is harmonically divided by the point O, the conic, and the polar of O, as defined in Art. 7.

I shall deduce the proof from the corresponding property of plane conics.

Let O be a pole, and PR its polar; then, from any point of the polar as P, there will radiate a spherical pencil.



Project the spherical conic upon a tangent plane at P; and suppose O, A, C, and B to be projected into O′, A′, C′, and B′. It is evident, then, that O′ is the pole of PC′, and

$$\therefore \frac{\text{O}'\text{B}'.\text{A}'\text{C}'}{\text{O}'\text{A}'.\text{C}'\text{B}'} = 1.$$

This ratio, however, depends upon the sines of O′PA′, &c.; hence, since all lines on the plane are perpendicular to the radius drawn to P,

$$\frac{\sin \text{OPB} \sin \text{APC}}{\sin \text{OPA} \sin \text{CPB}} = \frac{\sin \text{OB} \sin \text{AC}}{\sin \text{OA} \sin \text{CB}} = 1.$$

32. Let $\text{OB} = \varrho_1$, $\text{OA} = \varrho_2$, $\text{OC} = \varrho$; then

$$\frac{\sin(\rho - \rho_2)}{\sin(\rho_1 - \rho)} = \frac{\sin \rho_2}{\sin \rho_1},$$

$$\therefore \frac{\tan \rho - \tan \rho_2}{\tan \rho_1 - \tan \rho} = \frac{\tan \rho_2}{\tan \rho_1},$$

$$\therefore \frac{1}{\tan \rho} = \tfrac{1}{2}\left(\frac{1}{\tan \rho_1} + \frac{1}{\tan \rho_2}\right).$$

From this can be found the equation of the polar of the origin. The general equation of a spherical conic, transformed to spherical polar co-ordinates by

$$x = \cos\theta \tan\varrho,\ y = \sin\theta \tan\varrho,$$

becomes

$$(a\cos^2\theta + 2h\sin\theta\cos\theta + b\sin^2\theta)\tan^2\varrho$$
$$+ 2(g\cos\theta + f\sin\theta)\tan\varrho + c = 0.$$

Then using $\tan\varrho$ as the variable, we find the equation of the polar, by the same process as in plane conics, to be

$$gx + fy + c = 0.$$

From this equation, it is evident that the conic polar of the centre is the circle of which the centre is the spherical pole. This can also be proved by putting OA = OB in the last article. This circle corresponds to the line at infinity in plane conics.

33. The condition that three arcs meet in a point, and the equation of an arc through the intersection of two other arcs, are the same in spherical co-ordinates as for lines in plane co-ordinates. We can then prove, exactly as in plane conics, the following theorems: — Draw any two arcs through a point O; join directly and transversely the points where these arcs cut the conic. Then, if the direct arcs intersect in P, and the transverse in R, the arc PR is the polar of O.

The lines joining the corresponding vertices of a spherical triangle and its conjugate meet in a point.

If a quadrilateral be inscribed in a conic, each of the three points of intersection of the diagonals will be the conic pole of the arc joining the other two.

All of these properties are also seen to be true by projections on a plane.

34. The polar of $x'y'$ relatively to a conic is

$$\frac{xx'}{a^2} + \frac{yy'}{b^2} = 1,$$

of which the spherical pole is $\left(-\frac{x'}{a^2} - \frac{y'}{b^2}\right)$; the polar of $x''y''$ relatively to the supplementary conic is

$$a^2xx'' + b^2yy'' = 1;$$

and, if this is the polar of $\left(-\frac{x'}{a^2} - \frac{y'}{b^2}\right)$, it becomes

$$xx' + yy' = -1,$$

whose spherical pole is $x'y'$. Hence, to a point and its polar with reference to a conic, there correspond an arc and its conic pole with reference to the supplementary conic.

35. From Art. 30 can be deduced: If, from four fixed points on a spherical conic, arcs be drawn to a fifth point of the conic, their anharmonic ratio is constant. Reciprocally: If four fixed tangents be drawn to a conic, they will cut a fifth tangent in four points whose anharmonic ratio is constant.

36. If the spherical conic be projected upon a plane tangent to the sphere at the pole of a cyclic arc, the conic becomes a plane circle, and the cyclic arc a line at infinity. The plane passing through the centre of the sphere parallel to the plane of projection is the cyclic plane; and, if two planes be drawn through the centre of the sphere and through any two lines in the plane of projection, these two planes will intersect the parallel cyclic plane in two radii making the same angle as the lines. Since the centre of the circle is the pole of the line at infinity, its projection on the sphere will be the conic pole of the cyclic arc. By means of this method, many properties of the circle can be extended, with suitable modifications, to spherical conics. The propositions of Arts. 21 and 22 can be proved in this way, though in the inverse order.

37. If two tangents to a conic intercept upon a cyclic arc a segment of constant length, the locus of the point of intersection of these tangents is a second conic, and the arc joining the points of contact of the tangents will envelope a third conic. The cyclic arc will be a cyclic arc of the new conics, and will have the same conic pole for all three conics. For, if two tangents to a circle make a constant angle, the locus of the intersection is a circle, and the chord of contact envelopes a third circle, and these three circles are all concentric. Reciprocally: If a constant angle has its vertex at either focus of a spherical conic, the arc joining the points in which the sides of the angle cut the curve will envelope a second conic, and the tangents to the given conic at the points of cutting will intersect on a third conic, then (Art. 34) the focus at which the vertex of the constant angle is placed will be a focus for the three conics, and the directrix will also be the same for all three.

This example is sufficient to illustrate the method of applying the principle. It is plain that all graphic properties of the circle can be extended to spherical conics.

38. If now we suppose the radius of the sphere to become infinite, the spherical conics become plane. As the properties already proved still hold good, we can deduce the well-known properties of plane conics from corresponding ones of spherical conics. A remarkable analogy has been shown to exist between the foci and cyclic arcs; and, as the properties of the foci hold good in the plane conics, the question naturally arises, What becomes of the reciprocal properties of the cyclic arcs? In the case of the ellipse, the cyclic arcs become lines at infinity; but, in the hyperbola, the cyclic arcs become the plane asymptotes.

The Calculus can be applied to the equations of the spherical conics, and expressions can thus be found for the area and arc. As these, however, involve elliptic integrals, I have omitted them.